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A COMPARISON OF TWO METHODS FOR

SOLUTION OF THE PROBLEM OF BOLZA

E. WARREN SEIBEL

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This work is accepted as fulfilling
the thesis requirements for the degree of

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ABSTRACT

Two numerical methods of determining optimum paths in the problem of Bolza are presented. The basic theory of the methods is outlined and their essential characteristics are demonstrated by several specific applications to ship routing.

The first method is characterized as one of differential corrections employing methods developed by Professor Faulkner of the U. S. Naval Postgraduate School. The second method is termed a method of gradients employing calculation routines developed by Professor Bryson of Harvard. On the basis of the applications comparisons are made in the areas of ease of application and speed in producing results.

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INTRODUCTION

The problem of obtaining maximum or minimum performance in a variety of controllable situations is undoubtedly as old as time. Usually the degree of success is determined by the efficiency with which some vital, limited quantity is expended.

Often the best solution cannot be defined, except as it develops by experience, even though the problem can be defined and stated mathematically. The magnitude of the analysis involved limits the feasibility of determining the solution. Modern high speed computing machinery has, for many problems, removed the latter limitation. As a result renewed interest has been shown and much progress made in the development and application of control theory to such problems of optimum control.

This paper presents an outline of two practical numerical methods of solution of some general problems of this type. The class of problems is more specifically termed the problem of Bolza [1]. For purposes of identification in this paper the two methods will be called the 'differential method' and the 'method of the fundamental lemma' which will henceforth be abbreviated the D-method and the FL-method respectively.

The D-method is based on the formulas for differentials due to Bliss [2]. The numerical scheme for determining the constants of the solution by a Newton iteration is the work of

Faulkner [3],[4]. His research papers and related studies are the source of both the theory and the numerical data presented for the D-method.

In the literature the FL-method is also referred to as the 'gradient' and the 'steepest ascent' method. The calculational procedure is similar to the proof of the fundamental lemma and is apparently conceptually due to Courant [5]. The numerical scheme, which makes possible its application to the class of problems being considered, is the work of Bryson and Denham [6]. A similar method is presented by Kelly [7]. The theory and basic procedure for application of the FL-method, as reported in this paper, is that of Bryson and Denham. The contribution of the writer lies in the development of specific convergence schemes, left undefined by the authors [6], which make possible the programming and solution of two relatively simple ship routing problems thereby demonstrating several features of the method. Both problems have also been solved by the D-method and serve as a basis for limited comparison of the methods.

INTRODUCTION TO PROBLEMS IN OPTIMUM CONTROL

1.1 Formulation of a problem

We desire the solution to the set of ordinary nonlinear differential equations

(1) $\dot{x}_i = f_i[x_1(t), x_2(t), \ldots, x_n(t), p_1(t), \ldots, p_m(t), t]^*$

$$i = 1, 2, \ldots, n; \quad t_o \leqslant t \leqslant T$$

which is optimal, in the sense of maximizing the terminal value of a performance criterion

$$g(x_1, \ldots, x_n, t)_{t=T}^{**},$$

while simultaneously satisfying terminal constraints

(2) $h_k(x_1, \ldots, x_n, t)_{t=T} = 0$; $k = 1, 2, \ldots, N; \quad 0 \leqslant N \leqslant n.$

To effect these conditions we have at our disposal m control or driving functions $p_s(t)$, s = 1, 2, ..., m. Their definition for $t_o \leqslant t \leqslant T$ is a major portion of the problem.

---

* A dot ($\dot{}$) over a variable is used to designate the derivative with respect to t.

** If an integral is to be maximized we introduce an additional state variable, x_{n+1}, and an additional differential equation

$$\dot{x}_{n+1} = f_{n+1}(x_1, \ldots, x_n, t)$$

where f_{n+1} is the integrand of the integral.

It is assumed the functions f_i are defined over an open region and are class C". It is also assumed the functions g and h_k are independent and of class C' in the terminal region. The variables $(x_1, x_2, \ldots, x_n)$ will be termed the state variables. All are functions of the independent variable t. The terminal value, T, of t is either specified or otherwise determined from a terminal condition. The values $(x_1, x_2, \cdots, x_n)_{to}$ are assumed specified.

It will be convenient to represent these sets of variables by matrices or by vectors. We define

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} \; ; \; p = \begin{bmatrix} p_1(t) \\ p_2(t) \\ \cdot \\ \cdot \\ p_m(t) \end{bmatrix} \; ; \; h = \begin{bmatrix} h_1(x,t) \\ h_2(x,t) \\ \cdot \\ \cdot \\ h_N(x,t) \end{bmatrix} \; ; \; f = \begin{bmatrix} f_1(x,p,t) \\ f_2(x,p,t) \\ \cdot \\ \cdot \\ f_n(x,p,t) \end{bmatrix}.$$

No distinction will generally be made between an n×1 (column) or a 1×n (row) matrix and a vector. If a quantity is best considered a vector, as in a scalar product, it will be designated, for example, $\vec{p}$.

Equations (1) and (2) then become

(3) $\dot{x} = f(x,p,t)$

(4) $h(x,t)_{t=T} = 0.$

We desire the solution of system (3) which maximizes the performance quantity $g(x,t)_{t=T}$ subject to the constraints (4). For purposes of introduction of terminology and some basic theory a brief review of the related elements of the Calculus of Variations is first presented.

1.2 The Classical Calculus of Variations

The problem formulated in section (1.1) is typical of those considered in the variational calculus - those of

determining maxima or minima or, in general, stationary

values (defined on page 8) of functionals such as g(x,t),

by seeking the argument functions x(t) and p(t) for

which the functional assumes the stationary value or ex-

tremum in question.

An outline of some phases of the classical method of

the variational calculus, as applied to a greatly simpli-

fied optimum problem, will serve to introduce terminology

and to present essential concepts which are the basis for

both methods of solution being considered. The method may

be termed 'indirect' in that it is based on the reduction

of the variational problem to differential equations the so-

lution of which, subject to the boundary conditions, is the

solution to the problem. The two methods considered in

this study may be termed 'direct' in that they consist of

construction of a sequence of functions that converges to

the desired function.

Following Courant and Hilbert [8], we consider the

simplest problem of the variational calculus. Find the

function y(t) which is such that $y(t_o) = A$ and

$y(t_1) = B$, Fig. 1, and the functional

$$F[y(t)] = \int_{t_o}^{T} f[t,y(t),y'(t)] \, dt$$

is maximized. Any function $y = \overline{y}(t)$ which meets the end

conditions, is of class C, and has a piecewise continuous

derivative is termed admis-
sible. We assume that there
is a class of admissible argu-
ments and that among them there
exists the solution to our
problem, the function y=y(t),
which maximizes (or minimizes,
hereafter we will refer only
to maxima) the functional
F[y(t)]. Suppose we have
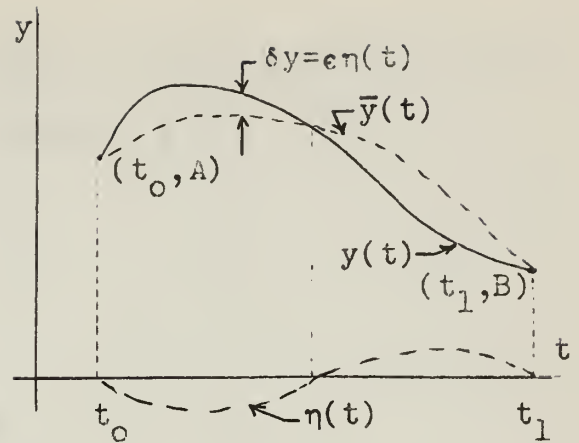
Fig. 1 An admissible
variation of an admissible
curve

found the desired extremal function y = y(t). Let us con-
sider also an unspecified new function $\eta(t)$ which is de-
fined for $t_o \leqslant t \leqslant t_1$, possesses the necessary derivatives
and continuity, and is such that $\eta(t_o) = \eta(t_1) = 0$, but is
otherwise arbitrary. The function

$$\bar{y} = y(t) + \varepsilon\eta(t) = y(t) + \delta y(t),$$

where ε is a parameter, represents a one-parameter family
of admissible functions which contains the solution y(t)
when $\varepsilon=0$. The quantity $\delta y(t) = \varepsilon\eta(t)$ is the variation
of the function y = y(t). For sufficiently small magni-
tudes of ε the varied functions $\bar{y}(t)$ lie in an arbi-
trarily small neighborhood of the extremal, y = y(t), and
the integral

$$F(\bar{y}) = F[y(t) + \varepsilon\eta(t)]$$

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may be regarded as a function $\Phi(\epsilon)$ which must have a maximum at $\epsilon = 0$ and therefore requires $\Phi'(0) = 0$.

Now

$$\overline{y} = y(t) + \epsilon\eta(t)$$

implies

$$\overline{y}'(t) = y'(t) + \epsilon\eta'(t)$$

and we may write

$$\Phi(\epsilon) = \int_{t_o}^{t_1} f[t,\ y+\epsilon\eta,\ y'+\epsilon\eta']\ dt$$

For a maximum it is necessary that

$$(5) \qquad \Phi'(0) = \int_{t_o}^{t_1} \eta[\ f_y - \frac{d}{dt}f_{y'}]\ dt\ = 0$$

for all $\eta(t)$ meeting the conditions previously imposed.

The fundamental lemma of the calculus of variations, [8] page 185, applied to (5) requires

$$(6) \qquad f_y - \frac{d}{dt}f_{y'} = 0,$$

or equivalently,

$$(7) \qquad y''f_{y'y'} + y'f_{y'y'} + f_{y'x} - f_y = 0.$$

This differential equation was first discovered by Euler in 1744, [9] page 22, and is commonly referred to as the Euler equation. A solution to the Euler equation is called an <u>extremal</u>. Its validity is a necessary condition

for the existance of an extremum,[8] page 185. Being a
differential equation of the second order, two arbitrary
constants must be determined for a solution which meets the
boundary conditions, i.e., to make an extremal an admissible
extremal. Every solution of Euler's equation is an ex-
tremal of the maximum problem,[8]. Courant describes (6)
the variational derivative of F with respect to t and
terms its role here as analogous to that of the gradient in
ordinary maximum problems.

In the n+1 dimensional case, with coordinates

$$t, \; y_1(t), \; y_2(t), \; . \; . \; . \; , \; y_n(t)$$

and the functional

(8) $$I = \int_{t_o}^{t_1} f(t, \; y_1, \; y_2, \; ..., \; y_n, \; y_1', \; y_2',.., \; y_n') \; dt,$$

the Euler equations take the form

(9) $$\frac{d}{dt}f_{y_1'} - f_{y_1} = 0; \quad \frac{d}{dt}f_{y'_2} - f_{y_2} = 0; \quad ....\frac{d}{dt}f_{y_n'} - f_{y_n} = 0,$$

a system of the second order equal in number to the number
of unknown functions $y_i(t)$ to be defined. Any n+1 space
curve

$$y_1 = y_1(t), \; y_2 = y_2(t), \; ......, \; y_n = y_n(t)$$

which is a solution to the system (9) is an extremal and
furnishes a **stationary** **value** of the functional (8).

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The necessary condition of Euler is usually called the first necessary condition. There are also necessary conditions of Weierstrass, Legendre and Jacobi as presented, for example, in [1] chapter 1. These are not discussed here; the fundamental problem confronting an applied mathematician is that of finding a likely candidate for a solution.

1.3 Introduction to direct numerical methods

The problem as stated in section (1.1) is similar to that resulting in system (9). The state variables $x(t)$ of our functional $g(x,t)$ are defined by a set of n first-order differential equations and one or more control functions, or parameters, $p(t)$; t is the independent variable. Solutions to the related Euler equations will, to the extent of meeting the first necessary condition, define the functions $p(t)$ such that the resulting path $x(t) = 0$ will impart at least a stationary value to the functional $g(x,t)_T$. The proper choice of the arbitrary constants of these solutions will result in admissible extremals if they are chosen such that the terminal conditions $h(x,t)_T = 0$ will also be met. It will be assumed that the initial conditions $x(t)_{t=t_o}$ are specified constants.
Various indirect approaches to such a solution exist. A bibliography to indirect method studies is presented in the survey paper [10]. Analytical solutions of the Euler equations involved usually require idealizing assumptions

which limit their applicability in practical situations. Under more realistic assumptions a numerical attack on these equations is required. Some direct methods are presented in [8] page 174. In the following sections the essentials of two additional practical methods are outlined. As with all methods of solving Bolza's problem, Lagrange multipliers are employed producing equations equivalent to the Euler equations in terms of solutions to an adjoint system of differential equations. The concept was developed by Bliss [2] in connection with differential corrections in ballistics.

The D-method uses only extremals. Then, by an iterative correction procedure, obtains an admissible extremal out of the family of extremals.

The FL-method produces an extremal out of any likely solution to the original equations; part of the problem is in a subroutine to approach an admissible solution. The solutions to the Euler equations, producing the extremal, are approached iteratively by a gradient, or steepest-ascent technique.

Since all methods of solution involve solutions to the adjoint equations this concept will be outlined before the details of the two methods are presented seperately.

1.4 The adjoint equations

The problem as stated in section (1.1) was to determine

the solution to the system

(3) $\dot{x} = f(x,p,t)$

such that the terminal constraints

(4) $h(x,t)_T = 0$

are satisfied and the functional $g(x,t)_T$ is maximized.

Of primary interest is the relationship between variations $\delta p(t)$ in the control variables $p(t)$, at any value t, and the resulting terminal variations δx_T of the state variables. We also need the relationships between $\delta p(t)$, δh_T and δg_T.

Looking at (3) in component form, (1), the variational relationships for the state variables can be expressed as

$$\delta \dot{x}_i = \frac{\partial f_i}{\partial x_1}\delta x_1 + \frac{\partial f_i}{\partial x_2}\delta x_2 + . . \frac{\partial f_i}{\partial x_n}\delta x_n + \frac{\partial f_i}{\partial p_1}\delta p_1 . . . . \frac{\partial f_i}{\partial p_m}\delta p_m$$

$$i = 1, 2, ..., n.$$

In matrix notation this is

(10) $\delta \dot{x} = F \, \delta x + G \, \delta p$

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where

$$F(t) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdot & \cdot & \dfrac{\partial f_1}{\partial x_n} \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ \dfrac{\partial f_n}{\partial x_1} & \cdot & \cdot & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}_{n \times n} \quad ; \quad G(t) = \begin{bmatrix} \dfrac{\partial f_1}{\partial p_1} & \cdot & \cdot & \dfrac{\partial f_1}{\partial p_m} \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ \dfrac{\partial f_n}{\partial p_1} & \cdot & \cdot & \dfrac{\partial f_n}{\partial p_m} \end{bmatrix}_{n \times m}$$

$$\delta p = \begin{bmatrix} \delta p_1(t) \\ \cdot \\ \cdot \\ \delta p_m(t) \end{bmatrix} \quad ; \quad \delta \dot{x} = \begin{bmatrix} \delta \dot{x}_1 \\ \cdot \\ \cdot \\ \delta \dot{x}_n \end{bmatrix}$$

We will use the prime symbol to denote the transpose of a matrix, e.g., $\delta x' = (\delta x_1, \delta x_2, \ldots, \delta x_n)$.

Following Bliss [2] the system of linear equations adjoint to (3) is defined to be

(12) $\dot{\Lambda} = -F'(t)\Lambda$

where

$$\Lambda'(t) = [\lambda_1(t), \lambda_2(t) \ldots \ldots \lambda_n(t)]$$

is a matrix of n Lagrange multipliers, and

$$\dot{\Lambda}' = [\dot{\lambda}_1, \dot{\lambda}_2, \ldots \ldots, \dot{\lambda}_n].$$

The solutions of systems (3) and (12) are related by

$$\frac{d}{dt}[\Lambda'\delta x] = \Lambda'G \, \delta p;$$

hence[*]

(13) $$\left[\Lambda'\delta x \right]_{t_o}^{T} = \int_{t_o}^{T} (\Lambda'G \, \delta p) \, dt.$$

Since we are assuming x_{t_o} fixed $\delta x_{t_o} = 0$ in (13) and we have

(14) $$\left[\Lambda'\delta x \right]_{T} = \int_{t_o}^{T} (\Lambda' G \, \delta p) \, dt$$

This is the differential formula relating the terminal changes in the state variables and the variations in the control variables. It is termed the Fundamental Formula by Bliss [2]. It is also known as the Fundamental Adjoint Formula and as the one-dimensional form of Green's Theorem [11], [12].

1.5   Applications of the Fundamental Adjoint Formula

Some indication of the usefulness of the fundamental formula is displayed in the following observations.

1) If Λ is chosen as the specific solution to the adjoint system (12) such that at $t = T$

$$\vec{\Lambda}_T = \nabla_x x_1(T) = (1,0,0,\ldots,0),$$ [**]

[**] $\nabla_x x_1$ refers to the gradient of x_1 in x space.

[*] We assume no corners, [9] page 8, exist.

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let us call this solution $\vec{\Lambda}_1(t)$, (14) then becomes

$$\delta x_1(T) = \int_{t_o}^{T} \Lambda_1' \, G \, \delta p \, dt \, ,$$

yielding the terminal variation of x_1 due to variations $\delta p(t)$. Similarly choosing $\vec{\Lambda}_i(t)$ such that

$$\vec{\Lambda}_i(T) = \nabla_x x_i(T) = (0,0,\ldots,1,\ldots,0)$$

$$i = 1,2,\ldots,n$$

gives

(15) $\qquad \delta x_i(T) = \int_{t_o}^{T} \Lambda_i' \, G \, \delta p \, dt \, .$

2) If Λ is chosen as the solution to the adjoint system such that $\Lambda_g(T) = \nabla_x g(T)$, (14) implies

(16) $\qquad \delta g(T) = (\nabla_x g \cdot \vec{\delta x})_T = \int_{t_o}^{T} \Lambda_g' \, G \, \delta p \, dt$

giving the first variation of the performance function in terms of the parameter variations for $t_o \leqslant t \leqslant T$.

3) For the constraint functions we likewise have

(17) $\qquad \delta h_j(T) = \int_{t_o}^{T} \Lambda_{h_j}' \, G \, \delta p \, dt \, , \quad j = 1, 2, \ldots, k.$

4) If we were to ignore the problem of meeting constraints and look for a solution which gives a stationary value for $g(T)$, we would require the vanishing of the first variation, (see appendix of [4])

(18) $\qquad \delta g(T) = \int_{t_o}^{T} \Lambda_g' \, G \, \delta p \, dt$

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which requires $\Lambda'_g G = 0$. That is

$$(\lambda_1, \lambda_2, \ldots \ldots, \lambda_n)_g \begin{bmatrix} \dfrac{\partial f_1}{\partial p_1} & \cdots\cdots & \dfrac{\partial f_1}{\partial p_m} \\ \cdot \\ \cdot \\ \dfrac{\partial f_n}{\partial p_1} & \cdots\cdots & \dfrac{\partial f_n}{\partial p_m} \end{bmatrix} = 0$$

which represents the m equations

(19) $\qquad \Lambda'_g \cdot \dfrac{\overrightarrow{\partial f}}{\partial p_i} = 0. \quad i = 1, \ldots \ldots \ldots, m.$

Equations (19) with the adjoint differential equations
(12) constitute what is generally termed the Euler-Lagrange
equations [4], [13].

In equations (18) the matrix $\Lambda'_g G$ is clearly an in-
fluence function giving for any time t the effect that a
unit impulse δp, at that time, would have on the perform-
ance function g(T). In that regard if we write $\Lambda'_g G$ as
a vector, say $\overrightarrow{A}(t)$, (18) then can be written

$$\delta g(T) = \int_{t_o}^{T} \overrightarrow{A}(t) \cdot \overrightarrow{\delta p} \, dt \qquad .$$

and for maximum change in the performance quantity g(T)
we would choose $\overrightarrow{\delta p}$ parallel to $\overrightarrow{A(t)}$ for $t_o \leqslant t \leqslant T$.
It should be noted that $\overrightarrow{A}(t)$ and $\overrightarrow{\delta p}(t)$ are m-dimensional
vectors in parameter space (p). Courant, [8] page 223, re-
fers to vector $\overrightarrow{A}(t)$ as the gradient of the functional
g(t) in function space. When the optimum solution or path

in function space is obtained this gradient will vanish.

In the case of our general problem we have, in addition to the requirement of maximizing the functional $g(x,t)_T$, the necessity for meeting terminal constraints $h(x,t)_T = 0$. We will choose the vector $\vec{p}(t)$ in parameter space such that the path meeting the constraints yields the largest possible value for $g(T)$. We now consider two methods of solving the problem.

We have called $g(x,t)_T$ a functional. It is a function which becomes a functional since (x,t) represents the endpoint of a curve satisfying the given differential equations with initial point $x(t_o)$ given. The function g has become a functional in the sense that the point (x,t) depends upon the curve.

TWO NUMERICAL METHODS

2.1   The differential method

    Recalling the fundamental differential formula

(14) $\left[\Lambda' \delta x \right]_T = \int_{t_o}^T \Lambda'_G \; \delta p \; dt$,

the D-method is based on the Euler equation which states
that if the path is to furnish a maximum to some functional
$g(x,t)_T$, it is necessary that the coefficient Λ'_G of δp
in (14) vanish for some solution Λ of the adjoint system
for all $t_o \leqslant t \leqslant T$. This requirement, $\Lambda'_G = 0$, expanded
in terms of (11), implies that for all $t_o \leqslant t \leqslant T$ there
exists a $\Lambda(t)$ and $x(t)$ such that

$$\overrightarrow{\Lambda(t)} \cdot \overrightarrow{\frac{\partial f}{\partial p_1}} = 0$$

$$\overrightarrow{\Lambda(t)} \cdot \overrightarrow{\frac{\partial f}{\partial p_2}} = 0$$

(20)

$$\overrightarrow{\Lambda(t)} \cdot \overrightarrow{\frac{\partial f}{\partial p_m}} = 0$$

The problem becomes: determine $\Lambda(t)$ and hence the control
functions $p(t)$ such that (20), (3), (12) and (4) are satisfied.

    Assuming first that T is specified we proceed to out-
line the method of determining the Λ matrix. On page 14
a set of n solutions of the adjoint system (12) was

developed such that

$$\vec{\Lambda_i}(T) = \nabla_x x_i(T), \quad i = 1, \ldots\ldots, n.$$

The linear combination

$$(21) \qquad \lambda(t) = c_1 \Lambda_1 + c_2 \Lambda_2 + \ldots\ldots + c_n \Lambda_n$$

of these n solutions is also a solution to (12).

As a first approximation to the admissible extremal solution desired we estimate a set of constants c_i for (21) and determine the associated extremal by employing this $\lambda(t)$ in system (20). This will in general not be an admissible extremal as it will not meet the terminal constraints (4).

On the basis of the error in meeting the terminal conditions, corrections dc_i for the constants in (21) are determined. The resulting improved $\lambda(t)$, when employed with (3) to obtain a new solution, will, if the method converges, give a more nearly admissible extremal. The process is repeated until some convergence criterion is satisfied.

If the terminal T is not specified it too will be estimated for the first solution and a correction dT will be produced with the dc_i at the end of each iteration.

A more detailed analysis of the basis and the calculation procedure is presented in appendix (A).

2.2 The method of the fundamental lemma

This method is concerned directly with the construction of the set of m control variables $p(t)$, for $t_o \leqslant t \leqslant T$, which produce the solution to (3) that satisfies the terminal conditions (4) and maximizes the performance quantity $g(x,t)_T$. Essentially the procedure is to produce any reasonable trial solution by the choice of a nominal control program $\overset{*}{p}(t)$. The resulting solution will in general be neither admissible nor an extremal. That is, the constraints (4) are not satisfied and the Euler equations (19), a necessary condition, are not satisfied. A better solution can therefore be constructed. A variation $\delta p(t)$ is then produced such that the Euler equations are more nearly satisfied and the solution's deviation from that of an admissible extremal is reduced. This variation applied to the nominal $\overset{*}{p}(t)$ produces a better control program $\overset{**}{p}(t) = \overset{*}{p}(t) + \delta p(t)$. The procedure might be described as iteratively stepping a control vector $\overrightarrow{p}(t)$, defined in parameter space for $t_o \leqslant t \leqslant T$, in the direction of steepest ascent toward meeting constraints and toward meeting the first necessary conditions of an extremal. This direction is that of $-\nabla_p h$ and $\nabla_p g$ in parameter space, where we use the subscript (p) to denote parameter space. As an admissible solution approaches an extremal $\nabla_p g$ will tend to zero [6]. The direction of δp, for each point of a nominal solution, will be determined

by the direction of the gradients at that point. The
magnitude of $\overrightarrow{\delta p}$ at each point will be best chosen pro-
portional to the magnitude of the gradients at that point
in parameter space.

In appendix B the desired parameter correction program
$p(t)$ is derived and defined for $t_0 \leqslant t \leqslant T$ as

(22)
$$\delta p(t) = \pm G'(\Lambda_{g\Omega} - \Lambda_{h\Omega} I_{hh}^{-1} I_{hg}) \sqrt{\frac{(ds)^2 - dh' I_{hh}^{-1} dh}{I_{gg} - I_{hg}' I_{hh}^{-1} I_{hg}}} + G'\Lambda_{h\Omega} I_{hh}^{-1} dh .$$

The plus sign is used if $g(x,t)_T$ is being maximized.

One phase of the calculation procedure as outlined by
Bryson [6], and summarized in appendix B, was found to be
somewhat indefinite. It is that of determining the arbi-
trary constant $(ds)^2$ which governs the magnitude of the
step size $\delta p(t)$ by limiting the integral of these mag-
nitudes for $t_0 \leqslant t \leqslant T$ by the relationship (B-7). It
seems apparent that the larger the magnitude $(ds)^2$, within
the limit which allows the linearization (10), the more
rapidly our solution process will converge to the admissible
extremal. Accordingly the first restraint on $(ds)^2$ will
be imposing an upper limit $(ds_0)^2$ as estimated from the
nature of the problem in conformance with (10).

Further it is obvious that repeated application of
large magnitude variations $\delta p(t)$ will not allow fine ad-
justments toward the admissible extremal, as the solution

is approached, unless the magnitude $(ds)^2$ is further manipulated. We therefore desire an automatic scheme which will rapidly step the nominal solution toward the ultimate solution but will adjust itself to finer increments in the vicinity of the admissible extremal. The following discussion of Bryson's procedure is to develop a basis for the design of such an automatic convergence scheme.

The expression (22) defining the vector $\overrightarrow{\delta p}$ consists of two components which we define $\overrightarrow{\delta p_h}$ and $\overrightarrow{\delta p_g}$. We designate

(23) $\delta p_h = G' \Lambda_{h\Omega} I_{hh}^{-1} dh$

since it is essentially a vector in p space whose magnitude is proportional to the magnitude of the error in meeting the constraints resulting from the nominal solution. $\overrightarrow{\delta p_h}$ has the direction, essentially, of $-\nabla_p h$. Accordingly $\overrightarrow{\delta p_h}$ is a component of $\overrightarrow{\delta p}$ which produces a more nearly admissible solution. The desired correction toward an admissible solution may be large and may require larger values of $|\overrightarrow{\delta p}|$ than the upper limit, previously imposed, will allow. We give construction of an admissible solution first priority, however, and up to the limit (ds_0) demand that $\overrightarrow{\delta p}$ be designed to meet the constraints.

The first term of (22), involving the radical, is essentially a vector in p-space whose direction is normal to $\nabla_p h$ and, as nearly as meeting constraints will allow, is parallel to $\nabla_p g$. It specifies a component $\overrightarrow{\delta p_g}$ which causes

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progression toward maximization of $g(x,t)_T$ without disturbing the progress toward an admissible solution made by $\overrightarrow{\delta p_h}$. The term

$$\sqrt{(ds)^2 - dh' \; I_{hh}^{-1} \; dh}$$

is what remains of magnitude $(ds)^2$ for use as magnitude for a vector normal to $\overrightarrow{\delta p_h}$. If $\overrightarrow{\delta p_h}$ demands all of $(ds)^2$ this radical is zero and $\overrightarrow{\delta p} = \overrightarrow{\delta p_h}$. Progress will be made toward an admissible solution and the functional $g(x,t)_T$ will increase or decrease according to whether or not $\overrightarrow{\delta p_h}$ has a component in or opposite to the direction of $\nabla_p g$.

Accordingly, the scheme for the choice of the magnitude $(ds)^2$ will be to first provide, within the upper limit, a magnitude

$$| \; G' \Lambda_{h\Omega} \; I_{hh}^{-1} \; dh \; |$$

for attaining an admissible solution, and second, provide for a component toward the extremal, adjusted according to the estimated deviation from an extremal. We define this latter magnitude (Δ), the magnitude of $\overrightarrow{\delta p_g}$. We then require

(24) $\qquad (ds_o)^2 \geqslant (ds)^2 = \Delta^2 + | G' \Lambda_{h\Omega} \; I_{hh}^{-1} \; dh \; |$

The initial choice and automatic adjustment of the magnitude (Δ) is another problem. We want it as large as possible initially. This will be controlled by (24).

We want smaller magnitudes as the ultimate solution is
approached, automatically adjusted to the need. The con-
vergence criterion suggested by Bryson [6] will now be em-
ployed. He develops the fact that on an admissible solu-
tion the denominator of the radical of (22) is the direc-
tional derivative or gradient

$$(25) \qquad \frac{dg}{ds} = \sqrt{I_{gg} - I'_{hg} I^{-1}_{hh} I_{hg}}$$

in p-space. This must decrease as the extremal is approached.
This quantity is, however, in general greater than zero. It
is reasoned that if a relatively large (Δ) is repeatedly
applied to an admissible solution it will rapidly decrease
the quantity dg/ds of (25) and then will ultimately cause it to
pass through a minimum, greater than or equal to zero. If
at this point the magnitude (Δ) is not reduced subsequent
solutions will be over-corrected toward an extremal; trajec-
tories which oscillate about the solution to the problem.
We use this characteristic to trigger a reduction of (Δ)
to some fraction, say 0.2Δ, and step from there toward
the extremal in smaller steps. The minimum value for dg/ds
attained this time will be smaller and the same behavior will
signal the need for further reduction in (Δ). The process is
repeated until some lower limit on (Δ) or dg/ds, a conver-
gence criterion, is attained.

A flow chart of a convergence scheme is presented as
appendix C.

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APPLICATIONS AND COMPARISON

3.1   A basis for comparison

The performance of both methods was investigated for
two ship routing problems. Of primary interest were:

(a)   the problem of choice of starting para-
meters that result in convergence to a
solution.

(b)   the domain of these parameters that pro-
duces a solution.

(c)   the range of problem variations that can
be solved with an acceptable set of these
parameters.

(d)   the rate of convergence to a solution.

For each application the performance criterion was
defined as the functional

$$(26) \qquad g = \int_{t_o}^{T} [\, c + e^{-\frac{(x - t/100)^2 + 4y^2}{2}} \,] \, dt$$

which could be interpreted, for example, as a probability
of detection or as a measure of accumulated radioactive
fallout in an area where the intensity is defined by the
time and position function $\mathring{g}(x,t)$. This is essentially
the problem considered by Faulkner [4] and Cook [14].
Their programs for the D-method were employed as a source

of comparison data for that method.

More specifically the problems considered are:

Problem I: Determine the route from a point within an
area where $\dot{g}(x,t)$ is defined, to any point
at which $\dot{g}(x,t)$ is a prescribed tolerable
maximum, such that the accumulated fallout
$g(x,t)_T$ is minimized.

Problem II: Determine the route between two prescribed
points in the area where $\dot{g}(x,t)$ is defined
such that $g(x,t)_T$ is minimized.

For the D-method the calculation procedure of page 18
for these problems is reduced to the specification of a
pair of constants defined by Cook [14] as (λ_o, μ_o) and are
the two components of the initial adjoint vector, which are
unknown at the outset of computation.

For the FL-method the starting parameters are, as for
any problem, a nominal $p^*(t)$ and a magnitude (ds_o). For
these problems $p^*(t)$ is a bearing angle or heading and
could be any constant direction $p^*(t) = k$ radians, rough-
ly away from the point of maximum $\dot{g}(x,t)$, 'ground zero'.
From (B-7) an acceptable value for (ds_o) is determined by
$(ds_o)^2 = (\delta p)^2 T$, where T is approximately the total time
required and δp is the change in bearing, at any point,
that would reasonably meet the linearity requirements of
(10). As an example; estimating T as 100 with an allow-
able δp of 0.5 radians would give:

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$$(ds_o)^2 = (0.5)^2(100) = (5)^2.$$

3.2 Problem I

We consider the specific case of problem I with start-ing point (x_o,y_o) specified as $(0.3,0.1)$. For the FL-method the choice $dS_o = 5$ is reasonable and a logical choice for $p^*(t)$ is $p^*(t) = \pi/2$ for $0 \leqslant t \leqslant T$ or very roughly 'away' from $\overset{\circ}{g}(max)$ at $(0,0)$. For the D-method a reasonable choice (with the advantage of hind-sight) for (λ_o,μ_o) is $(0,-100)$ which is a vector roughly in the direc-tion of the negative gradient of $g(x,t)$ with a magnitude approximating the final time T.

In an effort to experimentally determine the intervals of convergence an interval of values containing these esti-mated starting parameters was explored for both methods. The results are summarized in Table I and Figures 2 and 3. For the FL-method the process converged for $0.1 \leqslant p^*(t) \leqslant 3.2$ radians. The starting parameter is therefore any bearing angle roughly 'away' from $(0,0)$ within about a 180° interval. In Fig. 2 path A depicts the op-timum route. For values in the vicinity of $3\pi/2$ the local minimum at the opposite side of the cloud (path B in the sketch) is developed. If headed in the direction $p^*(t)= \pi$, however, the local maximum will not be produced but the nearest minimum (path A) will be generated, while $p^*(t)= 3.6$ results in convergence to path B.

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Table I Starting parameters for problem 1.

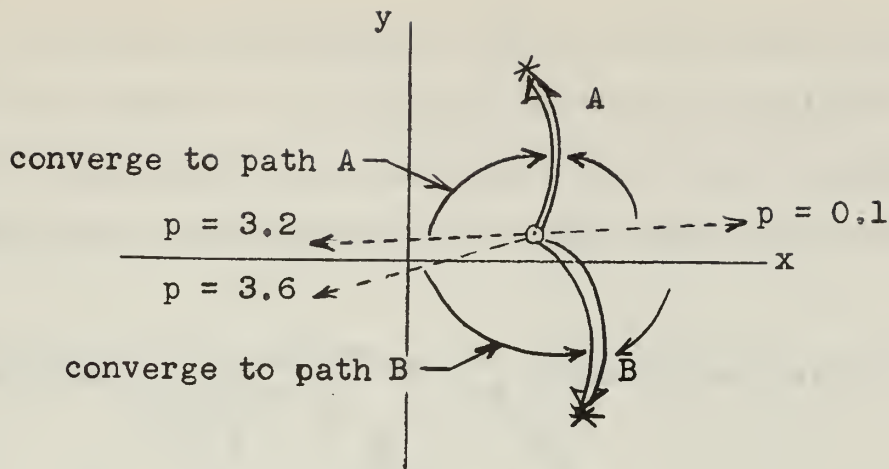| D-Method			FL-Method	
λo	μo	Convergence	$p^*(t)$	Convergence
90	-10	yes	0.1	yes
80	-20	yes	0.2	yes
60	-40	yes	0.4	yes
40	-60	yes	0.8	yes
20	-80	yes	1.2	yes
0	-100	yes	1.6	yes
-20	-80	no	2.0	yes
-40	-60	no	2.4	yes
-60	-40	no	2.6	yes
-80	-20	no	3.0	yes
-90	-10	no	3.2	yes
0	-50	no	3.6	(path B)
0	-200	yes		
60	40	(path B)		
80	20	"		
0	100	"		
-60	40	no		
-80	20	no		

Figure 2 FL Convergence Interval for $p^*(t)=k$

The number of iterations required for a 'solution' depends on the accuracy demanded. For values within the interval of convergence the poorest choice for $p^*(t)$ requires about ten percent more iterations than required of the best constant choice. No great penalty is associated with rough estimation of the starting parameter, for a close approximation to an admissible route is produced rapidly. Computer time is of the order of 45 seconds for 40 iterations. The first 15 iterations produced a very accurate solution.

For the D-method starting parameters were explored in the intervals $-100 \leqslant \lambda_0 \leqslant 100$, $-100 \leqslant \mu_0 \leqslant 100$. For purposes of probing the influence of magnitude $(\lambda_0, \mu_0) = (0, -50)$

and (0,-100) were also tried. Figure 3 and Table I sum-
marize the results. In each case for which a solution
existed convergence was relatively rapid requiring about
10 iterations and computer time in the order of 10 seconds
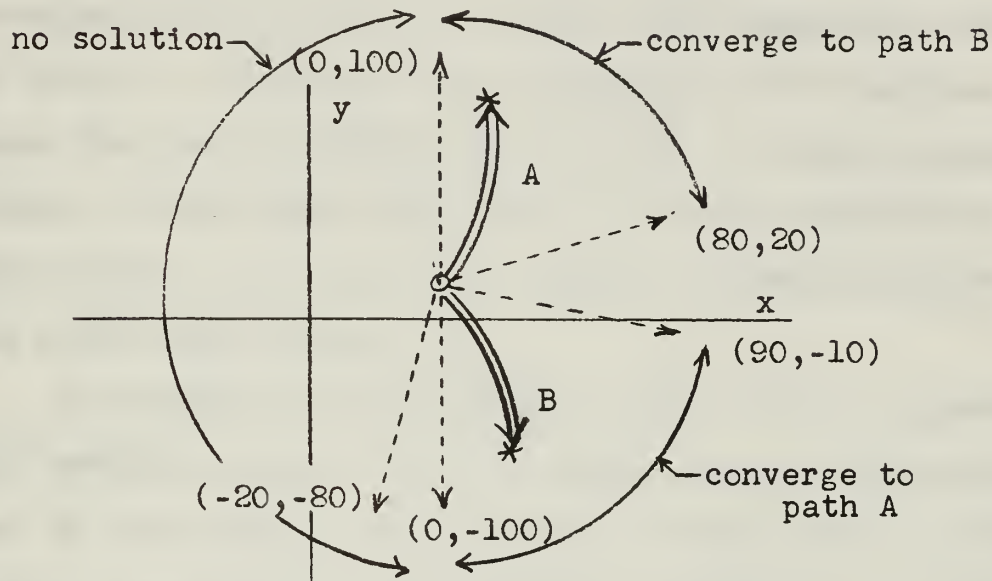
Figure 3 D-method Convergence Interval for (λ_o, μ_o)

for the particular convergence criterion employed.

Continuing with problem I we explore the domain of
starting points (x_o, y_o) for which a given set of starting
parameters produces a solution. For practical purposes it
would be desirable that a given set of parameters func-
tion for a large domain of starting points under the radio-
active cloud. Accordingly solutions were attempted for a
number of starting points in an interval bounded roughly by
the left and right extremeties of the intolerable fallout

area. The input parameters were maintained at $(\lambda_0, \mu_0) = (0, -100)$
for the D-method, and $p^*(t) = 1.0$ for the FL-method.

Figure 4 and table II portray the results for eight start-
ing points. Both methods produced solutions for each run. In
figure 4 the terminal points are associated with the corre-
sponding initial point by a dashed line simulating the route.
As table II indicates, the coordinates of the terminal points
agree for the two methods to the third or fourth significant
figure. Values for final time (T) and the performance g like-
wise agreed to at least four figures. Extended results for them
are given only for run D .

In regards to the FL-method, table III is a printout of
the terminal values of the iterations toward a solution for
run D of figure 4 and table II. It will serve to display
several features of the FL-method as applied to a relatively
extreme choice of the starting parameter $p^*(t)$. Specifically
the starting point is (0.3, 0.1) and $p^*(t)$ was, for probing
purposes, chosen as $p^*(t) = 0.1$. This is an obviously unwise
choice in as much as the cloud and the ship have comparable
speeds and, for this case, are almost parallel. Consequently
the terminal time (T) for the nominal run is about ten
times the value on the extremal. A nominal solution is pro-
duced, however, and from it relatively rapid convergence to
an extremal is portrayed.

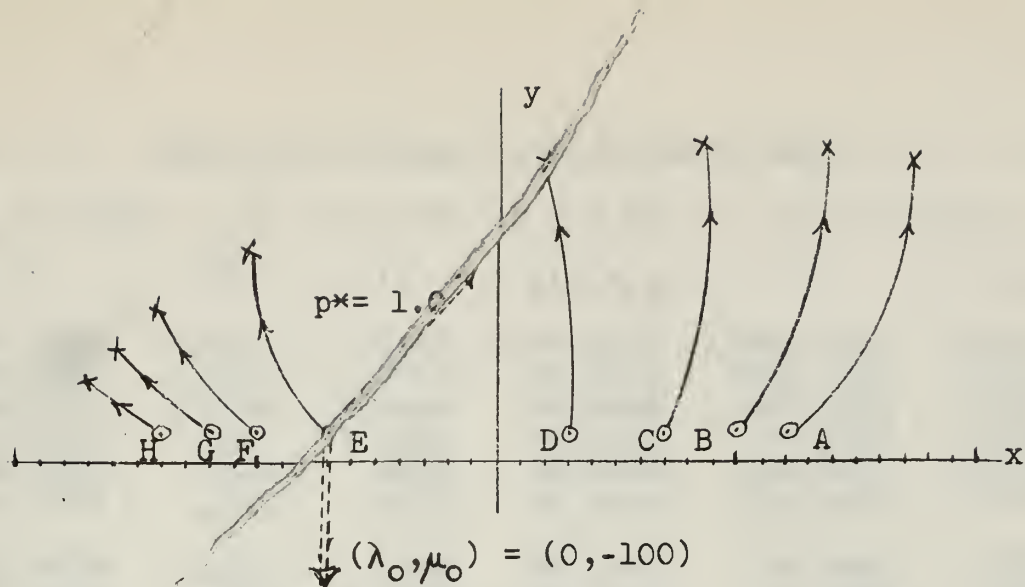The parameter (ds_0) was chosen as (6.0). The choice of

Fig. 4 Allowable Starting Points for (λ_o, μ_o) and $p^*(t)$ fixed

Table II Allowable Starting Points (x_o, y_o) for Problem I

$(\lambda_o, \mu_o) = (0, -100);$ $p^x(t) = 1.0$

Start point (x_o, y_o)	X(T)		Y(T)	
	FL	D	FL	D
A (1.2,.1)	1.4092	1.4103	1.0511	1.0510
B (1.0,.1)	1.1379	1.1388	1.0702	1.0702
C (0.7,0.1)	0.7266	0.7223	1.0661	1.0662
D (0.3,0.1)	0.1775	0.1778	1.0074	1.0075
E (-0.7,0.1)	-1.1066	-1.1071	0.6128	0.6125
F (-1.0,0.1)	-1.4175	-1.4179	0.4406	0.4401
G (-1.2,0.1)	-1.5879	-1.5882	0.3325	0.3320
H (-1.4,0.1)	-1.7292	-1.7293	0.2454	0.2452

Typical values for final time T and performance g(T)

for point D: $T_{FL} = 91.6399$, $T_D = 91.6417$

$g(T)_{FL} = 49.0500$, $G(T)_D = 49.0501$

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Table III Problem I, FL-Method, Run D

X_o = 0.3000 Y_o = 0.1000   S = 6.00 $p^x(t)$ = 0.10

T	X(T)	Y(T)	$g(X,t)_T$	D	ds
967.1896	9.9236	1.0656	487.8577	17661.3315	6.0000
256.7625	2.6762	1.0716	127.6031	2228.5757	6.0000
124.8307	1.0787	1.0696	62.3428	487.7631	6.0000
95.2305	.3507	1.0300	49.8881	79.2727	6.0000
98.7530	.1648	.9910	53.9286	275.8876	1.8000
94.0974	.1323	.9939	50.3492	132.3472	1.8000
91.8374	.1091	.9938	49.1760	29.3284	1.8000
92.2926	.2244	1.0146	49.3411	64.2659	.5400
91.8649	.2062	1.0121	49.1077	23.8514	.5400
91.6341	.1808	1.0080	49.0658	15.8866	.5400
91.7693	.1826	1.0081	49.1008	28.9107	.1620
91.6957	.1801	1.0078	49.0645	15.1434	.1620
91.6437	.1775	1.0074	49.0503	1.5992	.1620
91.6199	.1759	1.0072	49.0569	11.0875	.0486
91.6262	.1765	1.0073	49.0528	7.1449	.0486
91.6340	.1771	1.0074	49.0505	3.2402	.0486
91.6426	.1776	1.0074	49.0501	.7613	.0486
91.6293	.1766	1.0073	49.0503	3.1283	.0146
91.6331	.1769	1.0073	49.0501	1.9116	.0146
91.6372	.1773	1.0074	49.0500	.7058	.0146
91.6416	.1776	1.0074	49.0501	.4999	.0146
91.6376	.1773	1.0074	49.0500	.7230	.0044
91.6388	.1774	1.0074	49.0500	.3542	.0044
91.6399	.1775	1.0074	49.0500	.0208	.0044
91.6395	.1774	1.0074	49.0500	.3385	.0013
91.6396	.1774	1.0074	49.0500	.2254	.0013
91.6397	.1774	1.0074	49.0500	.1132	.0013
91.6399	.1774	1.0074	49.0500	.0149	.0013
91.6398	.1775	1.0074	49.0500	.0327	.0004
91.6399	.1775	1.0074	49.0500	.0116	.0004
91.6399	.1775	1.0074	49.0500	.0062	.0004
91.6398	.1775	1.0074	49.0500	.0189	.0001
91.6399	.1775	1.0074	49.0500	.0106	.0001
91.6399	.1775	1.0074	49.0500	.0031	.0001
91.6399	.1775	1.0074	49.0500	.0040	.0000
91.6399	.1775	1.0074	49.0500	.0000	.0000

Computer time: 44 seconds

any number $0 < ds_o < 20$ would also produce solutions. Actually values nearer 20 would hasten the convergence while values smaller than six would require a greater number of iterations.

Table III also portrays the operation of the convergence scheme discussed on page 23. Since in this case there are no constraints, Δ in (24) is (ds). As table III indicates, dS stays at its initially chosen value of (6.0) for four iterations. By this time the extremal has been closely approximated not withstanding the difficult nominal route chosen. For the fifth iteration dg/ds has crossed through its minimum and the reduction of (ds) to 0.3(dS) has been switched. The process then continued until an extremal was crossed again, causing a further reduction in step size. In this case the process was halted when the condition $dg/ds < 10^{-4}$ was attained.

3.3 Problem II

The two methods were less directly compared for problem II. Experimental probing was directed primarily at weaknesses of the convergence scheme devised for the FL-method. The first specific problem of type II was: Determine the optimum route from $(x_o, y_o) = (0.03, 0.1)$ to $(x_f, y_f) = (1.0, 1.0)$ with $c = 0$ in (26). An admissible solution, simulated by route (A) in figure 5, was closely approximated in the third iteration. Convergence to an admissible extremal, however, was not realized. Longer and longer admissible routes, (B), were developed, each better than the previous from the standpoint of

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decreasing the performance criterion, g . It then became appar-
ent that the solution to this problem does not exist. The quan-
tity g clearly would be minimized by escaping from under the
radioactive cloud to some very distant point, returning after
the cloud has moved away. The FL-method developed a sequence
of solutions which approached this solution. The D-method for
this case returned an'error' printout in as much as one of the
input parameters is a relatively close estimate of the final
time T. This value being undifined the process stopped due
to too large an error in the estimate of the magnitude of this
parameter.

The FL-method was then applied to the same problem with
c = 0.1 in (26). An admissible extremal resulted in a reason-
able number of iterations.

The importance of the careful choice of the stopping con-
dition $\Omega(x,t)_T= 0$, for the FL-method, was demonstrated by
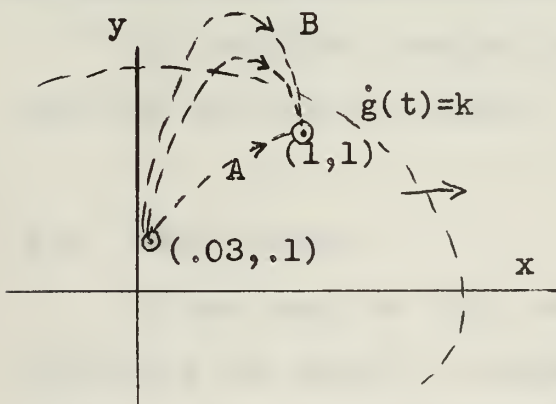problem II for the route (0.03,0.1) to (0.0,1.0), figure 6,
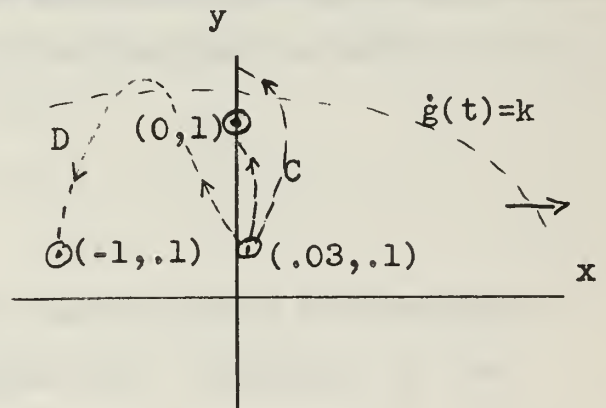

Figure 5 Problem IIa

Figure 6 Problem IIb,IIc

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with $\Omega(x,t)_T = 0$ chosen as $(x - x_F)_T = 0$. The resulting situation is that the stopping constraint is nearly parallel to the route making T, and thus $x(T)$ and $g(x,t)_T$ particularly sensitive to variations in the control variable. Consequently difficulty arose in that the constraint $(x - x_F)_T = 0$ was difficult to meet with the accuracy desired. The choice of the line $(y - y_F)_T = 0$, or the circle through (x_F, y_F) with center (x_o, y_o), would alleviate the situation. In general $\Omega(x,t)_T = 0$ and $\dot{g}(x,t)_T = 0$ should be as nearly parallel as possible.

The difficulty arising from too large a value for Δ_o was demonstrated for the FL-method by the solution of problem II for the route $(0.03, 0.1)$ to $(-1, 0.1)$, simulated by curve (D) of figure 6. The initial choices of magnitudes ten and five for Δ_o produced very slow convergence toward an admissible route. $\Delta_o = 0.1$, however, converged reasonably well. This can apparently be attributed to errors due to linearization on a path with relatively large curvature.

The D-method produced rapid convergence to solutions for each of the last two cases.

3.4 Conclusions

On the basis of this study the following observations regarding the points of comparison (a) through (d) of section 3.1 seem justified.

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(a)   The choice of acceptable starting parameters is distinctly easier for the FL-method. The matrix $p(t)$ is the set of actual control parameters for the problem and as such it is usually a set of well understood physical quantities. These are less abstract than the adjoint vector or the constants C_i of the D-method. General knowledge about the physical concepts involved usually permits estimation of a starting control program which relatively closely approximates that of the optimum solution.

(b)   The interval of convergence, i.e., the domain of starting parameters from which a solution can be produced, is significantly larger for the FL-method. Any reasonable estimate for the control parameter will start the solution. Table I and figure 3 indicate there are seemingly good estimates for (λ_o, u_o), such as $(-20,-80)$ which do not produce solutions for the D-method.

(c)   Once a workable set of starting parameters is found, however, both methods produce solutions from it for a wide range of variations of the problem.

(d) If the D-method converges it consistently results in the more rapid convergence. For similar convergence criteria the computation time ratio is about one to two, and the number of iterations required, a ratio of about one to four. The criteria employed were different for the two methods, however, and it appears from table III that the requirement $dg/ds < 10^{-4}$

was too tight for practical purposes. The precision is available, however.

The FL-method took only about ten percent more iterations to converge from a relatively wild estimate for $p^*(t)$ than it took for a very close estimate.

The following comments regarding the two methods are based on experience in implementing the two methods and on analyses of the papers by Faulkner [4] and Bryson and Denham [6].

The requirement of storing an entire control program and a complete set of functions $\Lambda'_{g\Omega}G$ for any one iteration is a disadvantage for the FL-method. This could result in memory storage problems for large problems or small computers. In contrast the D-method holds only the parameters C_i and T or their equivalent.

As indicated in Appendix B the FL-method has been developed to the point of providing for variations at $t=t_o$ and for the insertion of weighting functions by the matrix $W(t)$. The power of these features has not been investigated in this paper.

The problem of discontinuous control (corners) or unbounded control have also been ignored. The D-method has been developed to handle them, [4], while the application of the FL-method to such problems has not been undertaken, nor is it

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clear how this could be done.

With the D-method there is no question of how closely we have approximated an extremal, with the mathematical routine, since only extremals are used. The only limitation is the accuracy of the integration method. Also, since only extremals are used various identities associated with an extremal can be employed. For example, if the independent variable does not enter explicitly, the Hamiltonian is constant.

The integral relations for the corrections in the FL-method seem to be more stable from the computational standpoint, while, if the strong Legendre condition is not satisfied the routine must be modified in the D-method.

Only forward integration (or only backward integration) is used in the D-method. In the FL-method the basic equations are integrated forward and the corrections are obtained by backward integration of the adjoint system. This necessitates either storing the complete trajectory just produced or re-constructing it from terminal values by backwards integration of the basic differential equations of the system.

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A.1 Outline of the analysis

The following relations have been defined:

(3) $\qquad \dot{x} = f(x,p,t) \qquad$ the state variable equations.

(4) $\qquad h(x,t)_T \qquad$ N terminal constraints.

(12) $\qquad \dot{\Lambda} = -F'\Lambda \qquad$ the adjoint system.

(14) $\qquad [\Lambda'\delta x]_T = \int_{t_o}^{T} \Lambda' \, G \, \delta p \, dt \qquad$ the fundamental differential formula.

(21) $\qquad \lambda(t) = c_1\Lambda_1 + c_2\Lambda_2 + \ldots + c_n\Lambda_n = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$

where $\Lambda_i(t) = \begin{bmatrix} \Lambda_{1i} \\ \Lambda_{2i} \\ \vdots \\ \Lambda_{ni} \end{bmatrix}$ is the solution to (12) such that

$$\Lambda_{ij}(T) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and $\quad \lambda_i = \sum_{j=1}^{n} c_j \Lambda_{ij}$.

We write (14), using (21), as

(A-1) $\qquad [\lambda' \, \delta x]_T = \int_{t_o}^{T} \lambda'(t) \, G \, \delta p \, dt$

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From (20) and (21) we get the m equations

(A-2) $\lambda'(t) G(t) = 0$.

It will be recalled that (12) with (A-2) constitute the
Euler-Lagrange equations.

Additional relationships between the quantities dT,
δx_T, δp and dc_i will now be developed. See [4].
A differential dT in T results in the differential

$$dx_i(T) = \delta x_i(T) + \overset{\circ}{x}_i dT , \quad i = 1,2,\ldots,n ,$$

or

$$dx_T = \delta x_T + \overset{\circ}{x} dT$$

giving the variations

(A-3) $\delta x = (dx - \overset{\circ}{x} dT)_T$.

(A-3) with (A-1) gives

(A-4) $\lambda'(dx - \overset{\circ}{x} dT)_T = [\lambda'dx - \lambda'\overset{\circ}{x} dT]_T = \int_{t_o}^{T}[\lambda'(t)G(t)\delta p(t)]dt$.

If x is held constant in (A-2), which in expanded form is

$$\lambda'G = (c_1\Lambda'_1 + c_2\Lambda'_2 + \ldots\ldots + c_n\Lambda'_n) G(x,p,t) = 0 ,$$

we can write

$$(c_1\Lambda'_1\frac{\partial G}{\partial p} \delta p + \Lambda'_1 G dc_1) + \ldots\ldots(c_n\Lambda'_n\frac{\partial G}{\partial p} \delta p + \Lambda'_n G dc_n) = 0,$$

where we define the $m \times n$ matrix

$$\frac{\partial G}{\partial p} \delta p = \frac{\partial G}{\partial p_1} \delta p_1 + \ldots + \frac{\partial G}{\partial p_m} \delta p_m .$$

In terms of (21) this is the set of m equations

(A-5) $\qquad \lambda' \frac{\partial G}{\partial p} \delta p + \Lambda'_1 G \, dc_1 + \Lambda'_2 G \, dc_2 + \ldots + \Lambda'_m \, G \, dc_n = 0.$

Solving (A-5) for δp_1, $\delta p_2, \ldots$, δp_m and substituting into (A-4) we get n equations, in the $n+1$ variables $T, c_1, \ldots, c_n$, of the form

(A-6) $\qquad [\, \Lambda'_i dx = \sum_{j=1}^{n} I_{ij} dc_j + \Lambda'_i \dot{x} \, dT \,]_T$, $i = 1, \ldots, n$.

The differential of the constraint h_k is

(A-7) $\qquad dh_k = \left[\sum_{i=1}^{n} \frac{\partial h_k}{\partial x_i} dx_i + \frac{\partial h}{\partial t} k dT \right]_T$, $\quad k = 1, 2, \ldots, N$.

For admissible curves h_k is specified. The end point must therefore satisfy the relation

(A-8) $\qquad dh_k = 0$, $k = 1, 2, \ldots, N.$

In vector notation (A-7) and (A-8) may be expressed by

(A-9) $\qquad [\, \nabla h_k \cdot \vec{dr} \,]_T = 0$; $k = 1, 2, \ldots, N,$

where ∇h_k is a gradient in x, t space.

For a maximum value of $g(x, t)_T$ we similarly have

(A-10) $\qquad [\, \nabla g \cdot \vec{dr} \,]_T = 0$

in x, t space indicating the differential of the terminal

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point must be such as to result in displacements (geometrically speaking) normal to the ∇g in x,t space, that is, tangent to the surface $g(x,t)_T = g(maximum)$.

Since the functions $g(x,t)$ and $h(x,t)$ are assumed to be independent the $N+1$ by $n+1$ matrix of coefficients from (A-9) and (A-10)

$$(A-11) \qquad M = \begin{bmatrix} \dfrac{\partial h_1}{\partial x_1} & \dfrac{\partial h_1}{\partial x_2} & \cdots & \dfrac{\partial h_1}{\partial x_n} & \dfrac{\partial h_1}{\partial t} \\ \vdots & & & & \\ \dfrac{\partial h_N}{\partial x_1} & \dfrac{\partial h_N}{\partial x_2} & \cdots & \dfrac{\partial h_N}{\partial x_n} & \dfrac{\partial h_N}{\partial t} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} & \cdots & \dfrac{\partial g}{\partial x_n} & \dfrac{\partial g}{\partial t} \end{bmatrix}$$

has rank $N + 1$.

The transversal condition, [4] and [2] page 196, may be stated as the condition that the rank of the $N+2$ by $n+1$ matrix

$$(A-12) \qquad M^+ = \begin{bmatrix} & & M & & \\ \lambda_1 & \lambda_2 & \cdots\cdots & \lambda_n & -\lambda'\mathring{x} \end{bmatrix} \qquad t = T$$

be $N+1$ and the rank of the matrices obtained by omitting either of the last two rows also be $N+1$. Row number $N+2$ of M^+ is the row of coefficients of dx and dT of (A-4). These conditions on (A-12) yield $n-N$ relations involving the terminal values of x, λ, and t which must be satisfied by a solution which gives at least a stationary value to $g(x,t)_T$. We designate them

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(A-13) $H_s(x, \lambda, t)_T = 0,$ $s = 1, 2, \ldots \ldots, n-N.$

The differential relations derived from (A-13), at the terminal point, are

$$(A-14) \quad \left[dH_s = \sum_i \frac{\partial H_s}{\partial x_i} dx_i + \sum_i \sum_j \frac{\partial H_s}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial C_j} dC_j + \left(\sum_i \frac{\partial H_s}{\partial \lambda_i} \dot{\lambda}_i + \frac{\partial H_s}{\partial t} \right) dT \right]_T$$

$$s = 1, 2, \ldots \ldots, n-N,$$

where λ_i is defined as in (21).

Equations (A-6), (A-9) and (A-14) are $2n$ equations in the $2n+1$ differentials dx_i, dT and dC_j, i and $j = 1, \ldots, n$.

A.2   Calculation procedure

(a) A solution of the problem is characterized by the $n+1$ parameters $T, C_1, C_2, \ldots C_n$. We arbitrarily choose one relation among the C_i. We might choose $\sum_i (C_i)^2 = 1$ or just designate one, say $C_n = 1$. We must be careful only that the designated C_i does not turn out to be negative or zero. We then guess a set of values for the remaining parameters and compute simultaneously a solution to the original differential equations (3), the adjoint system (12) and the Euler equations (A-2) or (20). This yields an extremal E_o which does not, in general, satisfy (4) or (A-13).

(b) Simultaneously calculate the correction integrals (A-6).

(c) Treat the differential relations (A-5) through (A-13) as differences evaluated at the end point of E_o. We then have

(A-15) $\Delta h_i = h_i - h_i(t,x)_T$, $i = 1, \ldots\ldots, N$

$\Delta H_i = -H_i(T)$, $i = N+1, \ldots\ldots\ldots\ldots, n$.

(d)   Solve for the n differences $\Delta C_1, \ldots, \Delta C_{n-1}$, ΔT and apply to the previous estimate for these parameters. If the original estimate was close enough the new parameters will result in a more nearly admissible extremal.

(e)   The process is repeated until the sequence of extremals converges to an admissible extremal within the limit of some convergence criterion. Additional analysis is required to distinguish between maxima, minima and stationary values.

Appendix B THE FL-METHOD

B.1 Outline of analysis

We have previously defined the relationships

(3) $\qquad \mathring{x} = f(x,p,t) \qquad$ n state variable equations

(4) $\qquad h(x,t)_T = 0 \qquad$ k constraints

(10) $\qquad \delta\mathring{x} = F\,\delta x + G\,\delta p \qquad$ variational relationships

(12) $\qquad \mathring{\Lambda} = -F'\Lambda \qquad$ adjoint differential equations

(14) $\qquad [\Lambda'\delta x]_T = \int_{t_o}^{T} \Lambda'G\,\delta p\,dt \qquad$ fundamental differential formula

Following Bryson[6] we define one of the functions of (4) as a stopping condition,

(B-1) $\qquad \Omega(x,t)_T = 0$

which produces a terminal value T for t, thus eliminating the need for guessing T.

In section (1.5) we developed, from solutions of the adjoint system,

(17) $\qquad \delta h_j(T) = \int_{t_o}^{T} \Lambda'_{h_j} G\,\delta p\,dt, \qquad j = 1,2,\cdots,k,$

or equivalently

(B-2) $\qquad \delta h(T) = \int_{t_o}^{T} \Lambda'_h\,G\,\delta p\,dt$

where $\Lambda_h(t)$ is defined as

$$\Lambda_h(t) = \begin{bmatrix} \lambda_{h_{11}} & \lambda_{h_{21}} & \cdots & \lambda_{h_{k1}} \\ \lambda_{h_{12}} & \lambda_{h_{22}} & \cdots & \lambda_{h_{k2}} \\ \vdots \\ \lambda_{h_{1n}} & \cdots\cdots\cdots & \lambda_{h_{kn}} \end{bmatrix}_{n \times k} .$$

We also developed

(18) $\delta g(T) = \int_{t_o}^{T} \Lambda'_g \; G \; \delta p \; dt$.

We now add, for (B-1), a similar relationship

(B-3) $\delta\Omega(T) = \int_{t_o}^{T} \Lambda'_\Omega \; G \; \delta p \; dt$

where Λ_Ω is the solution to (12) such that

$$\overrightarrow{\Lambda_\Omega}(T) = \nabla_x \Omega(T) ,$$

where ∇_x denotes a gradient in x-space.

For small perterbations the value of T will be changed only a small amount dT, so that

$$dg = \delta g + \dot{g}dT$$

(B-4) $dh = \delta h + \dot{h}dT$

$$d\Omega = \delta\Omega + \dot{\Omega}dT$$

where $\dot{g}, \dot{h}, \dot{\Omega},$ are defined as

$$\dot{g} = \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}f\right)_T$$

(B-5)
$$\overset{\circ}{h} = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}f\right)_T$$

$$\dot{\Omega} = \left(\frac{\partial \Omega}{\partial t} + \frac{\partial \Omega}{\partial x}f\right)_T \quad .$$

Accordingly (18), (B-2) and (B-3) modified for a change dT in T become

$$dg = \int_{t_o}^{T} \Lambda'_g \ G \ \delta p \ dt + \dot{g} \ dT$$

(B-6)
$$dh = \int_{t_o}^{T} \Lambda'_h \ G \ \delta p \ dt + \overset{\circ}{h} \ dT$$

$$d\Omega = \int_{t_o}^{T} \Lambda'_\Omega \ G \ \delta p \ dt + \dot{\Omega} \ dT \quad .$$

In order to limit the magnitude of the step $\vec{\delta p}$ such that the linearization (10) is reasonable Bryson imposes the limitation

(B-7)
$$(ds)^2 = \int_{t_o}^{T} \delta p' \ \delta p \ dT$$

where (dS) is a magnitude arbitrarily chosen as a reasonable upper limit to the integral (B-7).

Throughout this paper a simplified version of the FL-method, as developed by Bryson, has been employed in that variations at $t = t_o$ have not been considered and in that his matrix of 'weighting functions', $W(t)$, has been neglected.

Elimination of dT in equations (B-6) gives

(B-8) $\quad$ $dg = \int_{t_o}^{T} \Lambda'_{g\Omega} \ G \ \delta p \ dt$

(B-9) $\quad$ $dh = \int_{t_o}^{T} \Lambda'_{h\Omega} \ G \ \delta p \ dt$

where

$$\Lambda_{g\Omega} = \Lambda_g - \frac{\dot{g}}{\dot{\Omega}} \Lambda_\Omega \qquad \Lambda_{h\Omega} = \Lambda_h - \frac{1}{\dot{\Omega}} \Lambda_\Omega \ \dot{h}' \ .$$

From (B-7), (B-8) and (B-9) we form the linear combination

(B-10) $\quad$ $dg = \int_{t_o}^{T} (\Lambda'_{g\Omega} \ G - \nu' \Lambda'_{h\Omega} \ G - \mu \delta p') \ \delta p \ dt + \nu' dh + \mu \ (dS)^2.$

The second variation of (B-10) is

$$\delta(\delta g) = \int_{t_o}^{T} (\Lambda'_{g\Omega} \ G - \nu' \Lambda'_{h\Omega} \ G - 2\mu \delta p') \ \delta^2 p \ dt$$

For maximum dg the coefficient of $\delta^2 p = 0$ implies

(B-11) $\quad$ $\delta p = \frac{1}{2\mu} G' \ (\Lambda_{g\Omega} - \Lambda_{h\Omega}\nu).$

Eliminating ν and μ between (B-7), (B-9) and (B-11) we have

(B-12)

$$\delta p(t) = \pm^{*} G' (\Lambda_{g\Omega} - \Lambda_{h\Omega} \ I_{hh}^{-1} \ I_{hg}) \sqrt{\frac{(ds)^2 - dh' I_{hh}^{-1} \ dh}{I_{gg} - I'_{hg} \ I_{hh}^{-1} \ I_{hg}}} + G' \Lambda_{h\Omega} \ I_{hh}^{-1} \ dh$$

$*$ $\quad$ Use the plus sign for maxima.

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where

$$I_{hh} = \int_{t_o}^{T} \Lambda'_{h\Omega} \; G \; G' \Lambda_{h\Omega} \; dt \; ; \quad I_{hg} = \int_{t_o}^{T} \Lambda'_{h\Omega} \; G \; G' \Lambda_{g\Omega} \; dt$$

$$I_{gg} = \int_{t_o}^{T} \Lambda'_{g\Omega} \; G \; G' \Lambda_{g\Omega} \; dt$$

B.2 Calculation procedure for FL-method

1. State a reasonable nominal control variable program, $p^*(t)$ which is defined for the time period (t_o, T). In many problems this can be a constant program. For example if the parameters are two directions, say the vertical angle ϕ and the bearing θ, we could estimate $p^*(t)' = (\phi_o, \theta_o)$, where ϕ_o is any constant value such that $0 < \phi_o < \pi/2$ and θ_o is somewhere in the interval (correct heading $\pm \pi/2$). Usually better estimates can be made.

2. With $p^*(t)$ solve system (3) determining T and $x(t)$ for $t_o \leqslant t \leqslant T$.

3. Solve the adjoint system (12) backwards simultaneously building $\Lambda_{h\Omega}$, $\Lambda_{g\Omega}$, I_{hh}, I_{hg}, I_{gg}. Store $\Lambda'_{g\Omega} G$ and $\Lambda'_{h\Omega} G$.

4. On the basis of $x(T)$ from step two, evaluate $h(x,t)_T$. Set $dh = -h(x,t)_T$.

5. Select a reasonable value for dS. If the numerator of the radical in (B-12) is negative scale down dh such that the numerator is $\geqslant 0$.
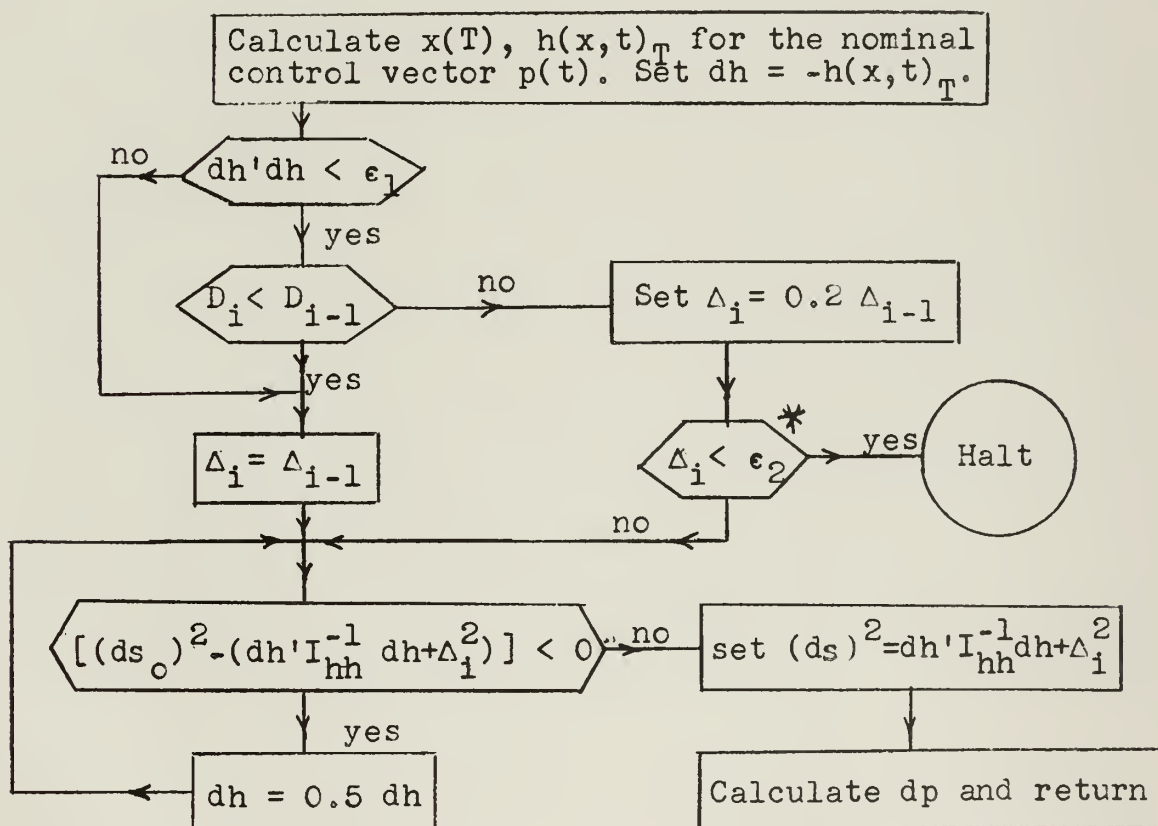
6. Produce $\delta p^*(t)$ from (B-12) for $t_o \leqslant t \leqslant T$ simultaneously forming $p^{**}(t) = p^*(t) + \delta p^*(t)$ and store for the next iteration.

7. Return to step two forming a new solution until the halt criteria are satisfied.

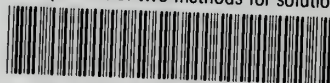Appendix C   Flow diagram for adjustment of (ds)

Definition of symbols

dh   -the error in meeting the constraints $h(x,t)_T = 0$.

ϵ_1 -the magnitude of dh within which it becomes

reasonable to be concerned about extremality.

ϵ_2 -the convergence criterion for extremality.

Δ_i -a part of (ds) whose magnitude is adjusted down as

the solution approaches an extremal.

D_i -the denominator of the radical of (B-12),

dg/ds in p-space.

Calculate $x(T)$, $h(x,t)_T$ for the nominal
control vector $p(t)$. Set $dh = -h(x,t)_T$.

no ← $dh'dh < \epsilon_1$

yes

$D_i < D_{i-1}$ → no → Set $\Delta_i = 0.2\ \Delta_{i-1}$

yes

$\Delta_i = \Delta_{i-1}$ $\Delta_i < \epsilon_2$ → yes → Halt

no

$[(ds_o)^2 - (dh'I_{hh}^{-1}\ dh + \Delta_i^2)] < 0$ → no → set $(ds)^2 = dh'I_{hh}^{-1}dh + \Delta_i^2$

yes

$dh = 0.5\ dh$ Calculate dp and return

* $D_i < \epsilon_2$ could also be used here.

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